Elementary Students’ Construction of Geometric Transformation Reasoning in a Dynamic Animation Environment

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> Context • Technology has not only changed the way we teach mathematical concepts but also the nature of knowledge, and thus what is possible to learn. While geometric transformations are recognized to be foundational to the formation of students’ geometric conceptions, little research has focused on how these notions can be introduced in elementary schooling. > Problem • This project addressed the need for development of students’ reasoning about and with geometric transformations in elementary school. We investigated the nature of students’ understandings of translations, rotations, scaling, and stretching in two dimensions in the context of use of the software application Graphs ’n Glyphs. More specifically, we explored the question “What is the nature of elementary students’ reasoning of geometric transformations when instruction exploits the technological tool Graphs ’n Glyphs?” > Method • Using a design research perspective, we present the conduct of a teaching experiment with one pair of fourth-graders, studying translation and rotation. The project investigated how and to what extent activity using Graphs ’n Glyphs can elicit students’ reasoning about geometric transformations, and explored the constraints and affordances of Graphs ’n Glyphs for thinking-in-change about geometric transformations. > Results • The students proved adept using the software with carefully designed tasks to explore the behavior of two-dimensional shapes, distinguish among transformations, and develop predictions. In relation to varied conditions of transformations, they formed generalizations about the way a shape behaves, including the use of referent points in predicting outcomes of translations, and recognizing the role of the center of rotation. > Implications • The generalizations that the students developed are foundational for developing an understanding of the properties of transformations in the later years of schooling. Dynamic technological approaches to geometry may prove as valuable to elementary students’ understanding of geometry as it is for older students. > Constructivist content • This study contributes to ongoing constructivism/constructionism conversations by concentrating on the transformation of ideas when engaging learners in activity through particular contexts and tools. > Key Words • Geometry, transformations, constructionist technologies.
Elementary Students’ Construction

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Young children already possess a dynamic spatial sense of shape, seeing shapes as malleable – and often provide “morphing explanations” (Lehrer, Jenkins & Osana 1998: 142) for shapes they identify as similar (similar in the sense of resemblance) – and using non-rigid transformations (pulling and pushing sides and vertices) to transform a shape to another shape. Children are able to identify congruency and similarity as early as pre-school (Olive et al. 2010) and have a motion perception of Gt by understanding rotations as turns, reflections as flips, and translations as slides (Hollebrands 2004). Consequently, we argue that the teaching and learning of Gt can begin as modeling a dynamic motion (Edwards 2003; Hollebrands 2003; Yaglom 1962) to explore “the mental or physical manipulation of geometric figures to new positions or orientations” (Boulter & Kirby 1994: 298) and then expanding these experiences to recognize that Gt act on all the points of a figure (eventually, for instance, understanding that translation can be applied to all points in the plane based on a specific direction and distance generally defined by the translation vector (Yanik 2014)).

Consequently, our goal was to create a new activity setting in which students could extend the Gt experiences that had been reported in the previous studies. Using new tools, we intended to perturb students’ understanding of Gt, and thus promote more sophisticated knowledge about the subject (Noss & Hoyles 1996). We considered the affordances of technology to make this possible through the design of technological tools that expand children’s contextual neighborhood, which, according to Dave Pratt and Richard Noss (2010), “captures the domain over which the idea has been encountered and found to be powerful by the child in explaining the on-screen behaviour” (Pratt & Noss 2010: 94). Our approach was influenced by the notion of “windows in thinking-in-change” (Noss and Hoyles 1996), in which the researcher examines thought processes by introducing new ideas and trying to understand how the thinker connects these notions with their previous knowledge. We considered the design of the “window” to be a fundamental element of the research process, and examined it through Pratt and Noss’s notion of “designing for mathematical abstraction,” which refers to the creation of a domain through which students’ “contextual neighborhoods can be refined and expanded” (Pratt & Noss 2010: 97).

Designing a “window” on students’ Gt reasoning

Existing middle- and high-school level studies on Gt show evidence of the power of digital technology to create richer opportunities for the study and learning of this concept. Laurie Edwards (1991, 1997) examined the types of generalizations on Gt that students made using Logo, finding that the feedback students received on their actions allowed them to generate conjectures about the effect of a transformation on an object, which were then confirmed or disconfirmed by the software. Additionally, the development of dynamic geometry software offered the potential to “build” geometric objects in the software that reflected a structure by connecting ideas and identifying relationships between them, leading to the notion of “figure” as a bridge between unrestrained drawing and the mental geometric idea (Laborde 1995). Geometer’s Sketchpad and Cabri-Geometry have been widely employed as ways to explore high-school students’ views of the nature of Gt on a plane, illustrating the power of the “figure” in the development of students’ Gt understanding (e.g., Hollebrands 2003, 2007; Laborde 2001). These studies were pioneers in demonstrating how the concept of transformations can be learned in a more dynamic and conceptual way, but they do not provide any information on how this concept can be explored in the earlier grades.

The work of Jere Confrey et al. (2010) offered an entry point to connect the notion of transformations to elementary students’ prior experiences. Graphs ‘n Glyphs (GnG) is a multi-representational,
dynamic microworld, developed as part of a longer-term project to study early development of geometric and trigonometric reasoning, that promotes student familiarity with GT as the basis for modeling motion via digital animation in two-dimensional space (Confrey et al. 2006; Confrey et al. 2010). Among the affordances of the environment include students becoming familiar with GT by reflecting, translating, rotating, and scaling shapes on the coordinate plane to complete puzzles. Students are able to direct GnG to perform specific single geometric transformations or sequences of transformations through the use of a timeline, part of the software’s animation sequencer (top portion of Figure 1). For example, having selected (or constructed) a two-dimensional object on a coordinate grid, a student calls up the transformation dialog box, and selects “translation” as the desired transformation (Figure 1). The student then enters the desired values for the magnitude (which students discover includes information about direction) of that transformation (Figure 1). Enactment of the entered transformation in the animation sequencer activates the motion of the object, thus providing immediate visual feedback. Students can reflect on the relationship between predictions about the motion due to a GT, the outcome of the motion animation, and subsequently can construct generalizations of the effects of GT in the plane. By double-clicking on the translation on the animation sequencer, the transformation dialog box re-appears, allowing the students to revise the parameter values for the transformation and re-test their conjectures.

Following a proposal by Judith Schwartz and Michal Yerushalmi (1993: 7) that “important mathematical ideas can be introduced early on in the mathematical education of all students if the introduction is done in the context of interesting and powerful exploratory environments,” we aimed to test the conjecture that offering students an environment in which they use GT to build animations would trigger students’ interest and engage them in a constructive activity to expand the contextual neighborhoods of their GT reasoning.

Studying students’ transformation-based reasoning

We designed an exploratory study to investigate the question “What is the nature of elementary students’ reasoning of geometric transformations when an instruction exploits the technological tool Graphs’n Glyphs?” In particular, we aimed to a) investigate how and to what extent activity using GnG can elicit students’ reasoning about GT and b) explore the constraints and affordances of GnG for thinking-in-change (Noss & Hoyles 1996) about GT. A series of teaching experiments (Cobb et al. 2003) was conducted, based on specially designed tasks using GnG. The initial interview plan and tasks were informed by the ideas embedded in the study of Confrey, Maloney, and colleagues (Confrey et al. 2006). The current article discusses the activity of one pair of students, Nate and Blake, from the fourth grade, who attended an elementary school in the eastern United States. The teaching sessions lasted 40 minutes each and occurred over twelve days.

Considering the limited time that the students were available for the teaching experiment, we had to constrain the specific numbers of GT that we could explore with them. The designed tasks presented to students included:

1. translations, which model a rigid motion (sliding) in which the figure remains congruent;
2. rotations, which also model a rigid movement but, in contrast to translations, keep a point fixed (point of invariance – center of rotation);
3. scaling (similarity transformation or dilation) to illustrate the change in the size of the shape after enlarging or shrinking; and, finally,
4. stretching (dilation in the direction of only one of the two coordinate plane axes) to prompt the students regarding what changes and what stays the same in this transformation in contrast to the previous ones.

Exploration of transformations in middle and high school often begins with a study of a reflection in a line or against an axis, followed by a rotation as a composition of two reflections when the lines intersect and/or a translation when these lines are parallel. Two issues led us to focus on translations, rotations, whole-number dilations (scaling) and single-dimensional stretches:

- Time constraints, along with research showing that students have difficulties with translation and rotation even when they reach high school (e.g., Hollebrands 2004).
- In translations and rotations, the negative sign (minus in the students’ terminology) becomes an indicator of direction opposite to the normal, based to some extent on their knowledge of subtraction.

Previous work with GnG with middle-school students (Maloney, Nguyen & Confrey 2008) revealed that multiplicative transformations involving reflections and dilations by fractional values involve issues that are complicated for middle school students, who tend to conflate the use of a factor of −1 with multiplication by positive fractional values in reflections and fractional dilations and stretches (shrinking). Further, U.S. fourth-grade students typically have not yet been introduced to negative integers, or to multiplication with fractions or negative numbers. The difficulty of these issues led us to forego these topics in this teaching experiment with elementary students. (Due to space limitations, the teaching experiment’s work on scaling and stretching is beyond the scope of the current chapter.)

In order to ensure that students had sufficient prior knowledge of the coordinate plane and angular measurement to perform the designated GT in GnG, students were presented a series of tasks calling for them to identify given points in the first quadrant of the coordinate plane and to determine movements that would be required to move one point to another. Instruction was confined to the first quadrant to avoid confusion with negative coordinate values. Then four different types of tasks were used to introduce translation and rotation. The first two tasks involved physical manipulation and paper-and-pencil grids. Similar tasks were used for scaling and stretching GT.

Table task

Students examined a paper shape (square, rectangle, or triangle) on the table, one at a time, and were asked to close their eyes for 10 seconds. Then they were asked...
a series of questions such as “What could I have done to the square/rectangle/triangle?” or “What changes and what stays the same when you do that action?” The purpose was to examine their initial knowledge of GT and to initiate a discussion focused on transformational invariants.

**Coordinate plane task**

“13” The paper shapes were placed onto and moved on a sheet of coordinate grid paper; the students were asked the same set of questions as previously. The intent was to help them leverage the additional information they recognized from working on the grid (namely, the use of coordinates and how these change when a shape undergoes a transformation).

**GnG introductory task**

“14” Students were asked to perform a specific GT in GnG using the transformation dialog box (see Figure 1). Students initially explored the way GTs are entered into the GnG transformation dialog box, using a task calling for predetermined movements of a single shape either in terms of translations or rotations.

**GnG matching tasks**

“15” Students were presented two copies of the same figure, A and B. They were asked to predict one or more GTs (including a sequence, if they found it necessary) needed to modify the location, orientation, and/ or size of figure A in order to superimpose figure A onto figure B (referred to as making Shape A “match” Shape B). These tasks required students to translate or rotate shapes in discrete ways. After the students had predicted the necessary transformations, they were asked to use GnG to enact the transformation(s) in the software environment, and to verify their prediction or modify their conjecture, using the visual feedback offered by the software. Figures 2 and 3 show an example of a matching exercise for translation and rotation.

“16” The GnG translation tasks presented to students were, in order, horizontal translations, then vertical, and finally, diagonal (i.e., non-zero horizontal and vertical changes). The rotation tasks increased in difficulty as well, depending on the position of the center of rotation (inside the shape, on the outline of the shape, and outside of the shape).

“17” Our task design aimed to build on students’ prior GT understandings and to assist them in expanding their contextual neighborhoods on the subject. We considered that students’ initial understanding of transformations for modeling motion would include thinking of the figure that resulted from a transformation as the original figure itself after having been physically manipulated.1 Our objective was to develop that thinking by identifying transformations as operations that act on figures as a whole, i.e., simultaneously acting in the same exact manner on every component of a figure, to elicit recognition that GT act on all the points of the plane (Hollebrands 2003). Typical GT tasks present two distinct figures (preimage and image) for transformations, which may mislead the students to think of the post-transformation figure as a completely new figure, with the original figure in its original position or size. The matching tasks aimed to bridge this gap by presenting two figures (preimage and image) but prompting the students to create an animation to act on (match) the original figure.
In this article, we present some episodes from the teaching experiment to provide examples of the generalizations articulated by the pair of students. We initially present students’ thinking of each GT during the paper-and-pencil tasks followed by a description of how this thinking was developed as they interacted with the software tasks. We analyze the generalizations students made while working with GnG in terms of “situated abstractions” – generalizations that students form in order to act in specific mathematical contexts, and which are embedded in the particular context of the actions (Hoyles & Noss 1992). For example, rotational designations in GnG were designed to be consistent with traditional angle designations on the unit circle, e.g., a positive angle corresponds to an anti-clockwise rotation, and a clockwise rotation requires designation of a “negative” angle; therefore we explored how students’ articulations of GT reasoning were meaningful in relation to the specific features of the GnG environment.

**Translations**

- **19** During the Table task students argued that a shape can slide “up or down, diagonal, or left and right.” After moving to the Coordinate plane task, they recognized that translation involves changing the coordinates of points on the figure, but not the shape and size of the figure. They also identified the location of the vertices in the initial and final positions of a translation, as well as the nature of the change in location. For instance, they determined that “the points are changing to a (7, 5) and a (11, 5),” and concluded that “the vertical line [coordinate] never changes. Only the horizontal line [coordinate] changes.” Students initial articulations show they considered that the figure resulting from a translation was the original figure itself after being moved, having changed its position on the grid (coordinates).

- **20** When they began working with GnG, students recognized that when the horizontal change of a translation is positive, the shape moved to the right. Subsequently, we prompted the students to predict how to move a shape eight spaces to the left (Excerpt 1).

**Excerpt 1**

Researcher (R): How can you use that to make the square move 8 spaces to the left?
Nate (N): I write 8 on the x [They performed that and it goes to the right.]
R: What did we do wrong?
N: Try minus this time.
R: What is minus?
N: Minus means go down. Try minus this time. Do you have a minus button? [He put −8].
R: Why do you think minus?
N: Because minus means back or take away.

**Excerpt 2**

N: We have to do horizontal and vertical, both. Why are there two boxes?
R: You need only one? Try only one.
N: 9 right and 4 up.
Blake (B): We need to go 4 up and 9 to the right.
B: Wait it’s 8
N: No it’s 10
B: The first one is 4 up and 8 right. [They use one box for both values].
R: You are using both values at the same time?
N: Yes it’s the easy way.
situated articulations suggest that students began defining, and distinguishing between, the magnitude and direction of the translation vector. They also began to identify the role of the negative sign contextually as specifying directionality of transformations, instead of only considering it in the context of addition and subtraction calculations.

**“The easy way”**

Subsequently, the students were challenged to match shapes that required diagonal translations. Although for diagonal translations the worksheets given to students provided two complete panels to permit separate panels for specifying the vertical and horizontal translations (i.e., two transformations), as shown in Excerpt 2, the students entered the horizontal and vertical values in just one of the panels (stating this is “the easy way”), regarding the second panel as superfluous.

The above shows that students were able to define the magnitude and direction of translation vectors in diagonal translations. Coordinating the change of two parameters \((x, y)\) as happening concurrently may be considered to illustrate a situated version of defining a translation vector as a linear combination of the individual component vectors (parallel to the \(x\) axis and parallel to the \(y\) axis) that makes the set of vectors linearly dependent.

**“The points have to match”**

In attempting to match the shapes, without prompting, students used referents (corresponding points on each shape) to determine the nature of the transformation needed in the mapping from the first task on GnG (Excerpt 3).

Although they used correct referents for the first task, they did not do so for the second. However, by trying different values (Figure 4), they concluded that initially they were not using the “correct points,” arguing, “You can take the points and match it up with the other points and see how much it travelled.”

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2] We refer to translation vectors for convenience in this article, and in terms of interpreting students’ situated understanding; however, vector terminology and symbols were not introduced to the students.

**Excerpt 3**

1. Match the star on the left to the star on the right.

R: How can we make this [left-side] star to move here [to star on right]?
N: You have to go plus on this horizontal way. I think it is going to be 7 plus. No, it’s 15.
R: How did you find that?
B: Because the point was at the 5 and I went 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 all the way up to this point.
N: You see this 5 down here. The star is on the 20. So we count from 5 to 20.
B: See the point right here? [the lower edge of Star A] I counted to this point [the lower edge of Star B] and I got to 15.

**Excerpt 4**

9. Match the star A to the star B in one transformation.

R: What is the relationship of these points? Why did you choose these specific points?
N: Because if we take this point [the top vertex of star A] and go to that point [the right vertex of star B] then the \(y\) axis coordinates would be wrong because this point is lower. You have to go from this top point to the other top point or from this point [the right vertex of star A] to this point [the right vertex of star B].
In subsequent tasks, we prompted the students to explain their idea of “matching points” in diagonal translations, as shown in the example of the two stars in Excerpt 4. They stated that if they try to match the top vertex of star A to the right vertex of star B, their animation would be wrong.

By experimenting with different values and using the feedback from the resulting translation animations, students noticed that even if they chose different corresponding points, the translation would remain unchanged. They argued that “The only thing you will have to look in order to make it right is that the points have to match,” recognizing that they only had to determine the distance between any pair of corresponding points, a situated recognition that any segment connecting corresponding preimage and image points is congruent to each other and to the translation vector. At this point, we introduced the term “reference points” in order to create a common language of how we talk about the corresponding “matching points” (Excerpt 5).

Students were able to attend to the direction and magnitude of these translations, which would have gone unexamined had they only used motions and surfaces that did not require attention to specific background (grid) position information. These articulations show situated versions of constructing and performing mappings to translate figures and recognized that translation as an operation that acts on figures as a whole by simultaneously acting in the same exact manner on every point within the figure.

Before discussing rotation, the students needed to have understanding of degree as a measure of angle in order to engage with the tasks. We therefore supported the development of this understanding by first exploring angles as wedges of a circle and then as formed by two rays and an endpoint. Students’ understanding of angle was extended to include angles-as-turns by interacting with GnG. After the examination of angles, the sequence of tasks followed the same structure as with the translation tasks.

During the Table task, students identified several key features of rotation, such as “[they can be] clockwise and anti-clockwise” and “[the shape] changes where [the direction] it faces. Its shape would be the same.” When the position of the center of rotation of a shape was changed from the center to one of the vertices of the shape, they gave the example of the Earth to explain the difference (Excerpt 6).

As the result of a translation in GnG, each point in the figure has changed in the same (additive) way in relation to its corresponding point in the shape’s prior position. In a rotated figure in GnG, correspond-

Figure 4 • Matching task on the top, students’ trials on GnG at the bottom (from left to right).
ing points change in relation to rotation around some other referent point (the *pivot* point in GnG). The single reference point (the center of rotation) is the same for all the points in the figure, and must be specified (unless the default center point of the shape is used). While working with the GnG rotational matching tasks, in order to achieve the desired rotation, the students needed to ensure that the center of rotation was correctly positioned, and to determine the angle for the desired rotation. The students first explored rotation in GnG through a series of prompts to perform rotations around centers of rotation located at various points on the figure. In the software environment, (counter-clockwise rotations have positive angle values, clockwise rotations negative), each group of students extended their generalizations about the role of the negative sign as reversing the (normal) direction of rotation. They explained that they used the minus sign “Because anti means ‘not’ and I thought anti-clockwise is ‘not’ so I tried it;” however, after trying their conjectures they realized that “minus is clockwise.” They used referent positions of figures to estimate the angle of rotation from one position to the next, arguing that a clockwise rotation would be \(-90\) (Figure 5).

“Nate argued that a counter-clockwise rotation could match the shape in Figure 5: “if we do the other way it would be 270.” When he was asked to explain his reasoning he argued that “Because you see that up to here it would be 90 and up to here 180 and up to here 270” showing how he divided the rotation into 90-degree intervals. He reasoned iteratively from the original positioning; it was not clear whether he understands a full circle to be 360 degrees.

“Excerpt 7

R: Are all of the points going to change?  
B: No.  
R: Which points are not changing?  
B: The nose point [the center of rotation was initially placed on the “nose” of the fish].  
N: No. I think all the points will change.  
B: No. See... the pin is there [at the nose point]. The pin is stopping that part. The middle allows it to go everywhere.  
R: OK, so the mouth, where the pin is, that point does not change. Now let’s change the pivot [as in the figure below]. What do you think it will happen?

B: Every point that you change it to it’s going to stay. Because the pin is stopping it. Every point you put the pin on, it’s going to stop that point.  
R: What is the role of the pivot in the rotation?  
B: It moves all the parts except that point.  
R: Why is it there?  
B: It has to make it in the same place to rotate.  
N: Where you put it, it makes it stick right there, so when you move it like...it rotates around the pivot.  
R: So you are saying that the fish rotates around the pivot.  
N: Yes. And if we take the pivot right there [outside of the fish] the whole fish will move.

“The nose point is not changing”

“The nose point is not changing”

The students formed generalizations about the angle of rotations as well as about the role of the pivot. They noticed, “Where you put it [the center of rotation], it makes it stick right there, so when you move it [the shape], it [the shape] rotates around the pivot.” When students were prompted to notice what changes and what stays the same in a rotation, they concluded that all the coordinate points of the shape change while the point of the pivot (center of rotation) remains invariant (Excerpt 7).

Figure 5 • Students’ use of referent points (left) and testing their prediction on GnG (right).
Students concluded that in a rotation, the path travelled by a shape around the center of rotation "goes exactly as a circle... It is like an invisible circle is surrounding this." Subsequently, we introduced the term "center of rotation" to prompt the students to notice the distance between the shape and its assigned pivot (Excerpt 8).

Up to this point, students could reason about the motion of turning and the role of the pivot but did not consider the relationships between the preimage and image points, on one hand, and the center of rotation, on the other. Therefore, we gave them more advanced matching tasks, which required moving the center of rotation outside of the shape. By experimenting with GtG, they recognized that the center of rotation was a position that can be assigned according to the needs of a particular task, and that it was possible to place the pivot "outside" the shape being transformed. For example, in the task presented in Excerpt 9, Nate recognized that one strategy for rotating shape A directly onto shape B was to place the center of rotation for shape A in the "center" of the space between the two stars.

The above examples illustrate that as students solved the rotation tasks, they constructed situated abstractions about the invariance of the center of rotation (for a given rotation), as well as about the distance between the shape and its assigned pivot, recognizing that corresponding preimage and image points are equidistant from the center of rotation.

Retrospective analysis

This study showed that students’ contextual neighborhoods for Gt reasoning could be extended. Instead of relying on only appearance during paper-and-pencil tasks (e.g., the shape stays congruent in translation and rotation, and becomes similar in scaling), it focused on specific properties of particular Gt as they worked in situated ways through activities incorporating GtG alongside carefully designed tasks and probing questions. For translations, students’ talked in situated language about the direction and magnitude of the translation, adopting the notion of the negative sign as an indicator of opposite change. They also discussed translation as mapping in the context of "matching points" or referents, recognizing that the transformation acts on all the points of the figure. For rotations, they extended their generalization about the negative sign as the opposite direction for a rotation, relative to the "normal" direction, and came to include situated accounts of invariance when they noticed that the center of rotation, and the shape (including its size) was invariant under the transformation. By exploring rotations further, they recognized that the points of the preimage and image are equidistant from the center of rotation.3

Although omitted due to space constraints, it is worth mentioning that for scaling (dilation) tasks, the students asserted that the shape grows bigger or smaller, that the coordinates of the shape change not additively but through multiplication (a "scaling factor" in GtG parlance) and recognized that the position and size of the final figure depend on both the location of the pivot (point of invariance) and the scaling factor.
This study showed that students progressed from defining the domain of transformation as selected points of the figure (e.g., "the points are changing to a (7, 5) and a (11, 5)") to later include all points of the figure (e.g., "I think all the points [of the figure] will change") and (possibly) even including all the points on plane as a domain by referring to points outside of the figure being transformed (e.g., "Because everything rotates around it [the pivot]"). Although this conclusion was not directly explored with the students, this study suggests the significance of the technological setting in the formation of these articulations, although they were less formal and most of the times are “situated,” rather than abstract or symbolic, can serve as foundational versions of the above concepts.

GnG acted as a constructionist medium in that it engaged the users in activity that facilitated the growth of their initial naive ideas about GT. Trying to “match” the shape in GnG acted as a window to the embedded mathematical reasoning, giving students iterative opportunities to generate new experience and to modify previous conceptions. Seymour Papert (1980) talked of “debugging” as the process through which the children search for what they made wrong and try to find a way to fix it. The immediate visual feedback provided by the animation environment played a significant role in this process: students routinely used the GnG visual feedback to modify their conjectures to match the shapes and form generalizations. Referring to dynamic geometry systems, Lulu Healy and Celia Hoyles (2001: 238) observed, “Students generally want to explain why certain phenomena are observed on the computer screen, especially if visual feedback is surprising.”

The features (including constraints) of the tools and of associated activities prompted the students to adapt their problem-solving strategies and their language. For instance, students recognized that they needed to use the negative sign to move their shape “backwards” or “downwards” during translation, as well as to perform a clockwise rotation.

While working with computer software, pupils adapt their strategies to its constraints and functioning mode, and the new meanings are generated by this continuous process of adaptation, thus constructing new knowledge. (Osta 1998: 130)

The work reported here suggests that by providing students with opportunities to investigate relationships between transformations and the geometric space in which those transformations are enacted, GnG has the potential to support dynamically the advancement of students’ thinking of GT. As Andrea diSessa argued, “This is the idea of microworlds, constructing artificial realities that intersect enough with students’ ideas that they can immediately begin to manipulate them, but whose ‘deep structure,’ if you like, leads inevitably beyond those initial perceptions and conceptions.” (diSessa 1988: 62)

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Open Peer Commentaries
on Nicole Panorkou & Alan Maloney’s “Elementary Students’ Construction of Geometric Transformation Reasoning in a Dynamic Animation Environment”

Documenting the Learning Process from a Constructionist Perspective
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> Upshot • This commentary assumes a constructionist perspective to discuss the choice of methods, conclusions and design goals that Panorkou and Maloney make in their study of students’ activities with the Graph ‘n Glyphs microworld.

How do constructionists measure learning?

1. Constructionism has been cast as both a theory of learning with artefacts and a theory for designing artefacts along with accompanying sequences (cf. Kynigos 2015). In this commentary, I first examine some of the main assumptions of this theory, and then discuss how they apply to the research on learning and design described in the study conducted by Nicole Panorkou and Alan Maloney.

2. Seymour Papert claims that constructionism was purposely named to draw attention to the similarities and areas of departure with constructivism (for a description of radical constructivism over the past forty years, see Riegler & Steffe 2014). In particular, Papert (1992) argues that both theories are based on the belief that knowing is a personal, constructive act. Both the teacher and the student create their own constructions of what is heard and discussed through their own experiential lenses. The implication of this assumption about learning is that knowledge cannot simply be conveyed in ready-made form from one person to another, nor can learning be assumed to be similar among people, even if they are taking part in the same conversation.

3. Papert states that the point of departure between the two theories lies in constructivism’s apparent privileging of abstract thinking. This conclusion is based on his observations that teachers chose to use Jean Piaget’s theory of stages as a way to gauge and guide age-appropriate teaching. Papert claims that, especially with regard to young children, the result of the stage theory is that schools have a “…perverse commitment to moving as quickly as possible from the concrete to the abstract thus spending minimal time where the most important work is to be done” (Papert 1993: 143). He also states that the use of tests to measure abstract learning does not represent the valuable learning-in-action that comes from play and situated engagement.

4. The question that this fork in epistemology highlights is: How should educational researchers document learning? Panorkou and Maloney argue that the use of the teaching experiment methodology provides the best way to probe students’ motivations as they are constructing in the “physical world,” such as making a sand castle or a Lego house, or playing in a microworld such as Graph ‘n Glyphs (GrG). The challenge with this type of work is that the array of possible meanings and interpretations is almost limitless, thus making it difficult for researchers to substantiate claims of any particular or “abstracted” learning occurring, or that the same learning could have occurred among the individual conversants.

5. In their article, Panorkou and Maloney provide short snippets of conversations that serve as examples of how two of the students acted with each other and with the microworld to make sense of their activities and reach their goals. In my view, the chosen examples are poignant and very provocative. However, as noted in the proviso above, I am not sure that the examples that are shared support some of the authors’ conclusions regarding what each of the two students “learned.” For example, in Excerpt 2, the students decide to combine vertical and horizontal translations in one step. This is an excellent and situated idea that would be a natural action when working in the physical world. However, the authors’ conclusion that this decision “[…] may be considered to illustrate a situated version of defining a translation vector as a linear combination of the individual component vectors […]” that makes the set of vectors linearly dependent” (§23) seems unsubstantiated. That is, the data provided does not present compelling evidence that either of the fourth grade boys had constructed any notion of linear dependence, or even how the combining of vertical and horizontal components relates to linear combinations.

6. A second example of a meaning-making exchange that was fascinating in its own right but not sufficiently documented to substantiate generalized learning occurs
in Excerpts 6 and 9. Nate describes the difference between rotations about a point at the center of the figure versus one outside by drawing an analogy with the Earth rotating around its axis versus the Earth rotating around the Sun. This is an ingenious connection, but does not appear to be assimilated by Ben, who (in Excerpt 9) believes that Nate is “crazy.” The authors do not seem to take this sentiment into account, or include Nate’s reaction to the visual feedback they observe. Instead, the authors claim that

The above examples illustrate that as students solved the rotation tasks, they constructed situated abstractions about the invariance of the center of rotation (for a given rotation), as well as about the distance between the shape and its assigned pivot, recognizing that corresponding preimage and image points are equidistant from the center of rotation. 8

The data suggest that perhaps Nate did conceptualize this notion of the center of rotation, but there is no indication that Ben had assimilated the Earth rotation analogy, let alone constructed a situated abstraction of it that would relate to the center of rotation being placed either inside or outside the object.

How do constructionists design for learning?

8 When designing microworlds to support students’ play, designers consider the affordances of the media. One would assume that the GnG software was designed to exceed the shortcomings of the “table tasks” described in the article. Certainly, both media can support the students’ generalization that shape and size are preserved under translations and rotations. But, ironically, they had to work with the program to assure that students viewed translations as actions on the preimage in order to dispel the idea that the preimage and the image were different objects. Other affordances of the program were more clearly elaborated. For example, one of the program’s clear benefits is its ability to execute transformations based on inputted parameters such as length and direction of vectors or degrees of rotation. This feature enabled the students to look inside the “black box” of transformations in order to accomplish their goals, and, in so-doing, make meaning of how the feature worked.

9 Using Papert’s contention that microworlds allow students to “something else” instead of math (which I take to mean what students would consider performing memorized algorithms that relate to concepts rather than working on a physical-world constructions such as building Lego bricks), my first design question is: What was the students’ goal? What was the “other activity” in which the students were imagining that they were engaging? The authors briefly mention that the grant that supported this work was designed to engage students in the creation of animations. If this was the case and the larger motivation, then were they setting their own goals, or were they working on tasks that the interviewers provided? So, we might ask, was the original study designed to focus on students’ learning of abstract concepts regarding transformation geometry in the context of animation? If so, it would have been fascinating to see whether their situated activities resulted in the creation of animations. And, in particular, what aspects of their activity within the microworld affordance system enabled the students to become bri-colours (as quoted by Papert, who was describing the work of Claude Lévi-Strauss) by acting and revising their situated activity using the tools at hand.

10 A second, related question is: Why was the timeline feature detailed in the introduction, but not featured in the students’ activity? Was this feature designed to be an affordance of the software that could support multiple steps in one execution, or was it part of the larger goal of creating animations?

In summary, Panorkou and Maldeney provide some interesting observations describing students’ situated activity and their discussions to make sense of it from a constructionist perspective. Using the students’ actions and answers as the unit of analysis provided great insight into how they were using some of the affordances of the software and making sense of their actions (although not learning the same things). Thus, for me, the value of this article lies in its utility to support further design of software and instructional sequences. The work also provides a starting point for these and other researchers to replicate and build on the sequence and students’ responses in order to propose a fully developed model to support students’ learning of transformational geometry — perhaps within the broader context of learning animation. The goal would be to create research that contains what Alan Schoenfeld (2000) claims are standards for research models in mathematics education: descriptive power, explanatory power, scope, predictive power, rigor and specificity, falsifiability, replicability, and triangulation.

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Reasoning in a Dynamic Animation Environment
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> Upshot • Parnorkou and Maloney describe how a dynamic animation environment, Graphs ‘n Glyphs, supported fourth-grade students’ understandings of translations and rotations. Two elements were critical in their teaching experiment: the design of the software and tasks. This commentary focuses on the decisions that they made and possible implications they had for students’ reasoning.

« 1 » The importance of geometric transformations in developing students’ understandings of geometry and function has been acknowledged for some time and this topic has been included in the K–12 mathematics curriculum in the United States for several decades (Coxford 1991; National Council of Teachers of Mathematics 1989, 2000; Sinclair 2008). In addition, the study of geometric transformations is amenable to the use of non-technological and technological tools (Coxford 1991; National Council of Teachers of Mathematics 2000). Nicole Panorkou and Alan Maloney share interesting insights from their research about the ways in which a dynamic animation environment, Graphs ‘n Glyphs (GnG), supported fourth-grade students’ understandings of translations and rotations. Two elements were critical in their teaching experiment: the design of the software program and the design of the tasks. My commentary will highlight important decisions the researchers made in the design of the software and tasks and discuss possible implications these had for students’ reasoning.

Software design
« 2 » When designing software for geometric transformations, one must carefully consider the ways in which teachers and students interact with the tool, the information the tool provides, modifications teachers and students can make, and a history of actions and how they are displayed (Underwood et al. 2005). In this regard, GnG does a great job, displaying a history with the timeline. The coordinate grid background is another critical and important feature. With the coordinate grid, students build on their experiences with number lines for adding and subtracting to consider horizontal, vertical, and, finally, diagonal translations. It was clear that students were able to use the grid to perform translations and rotations. However, it was not clear how this interface scaffolds students as they transition from thinking about translations performed in a vertical and horizontal direction to thinking about translations defined by a vector.

« 3 » To examine how other software programs represent vectors, Cabri Geometry II Plus, Geogebra, and The Geometer’s Sketchpad were reviewed. Cabri Geometry II Plus requires the selection of the object and then the selection of the vector that defines the translation.

« 4 » Geogebra requires users to interact with the software in a similar way as Cabri. The Geometer’s Sketchpad allows one to translate by a polar, rectangular, or marked vector (Figure 1).

« 5 » What is interesting with this interface is that students are able to enter a negative value for the horizontal or vertical fixed distance to move left or down. Although the GnG interface is most closely related to The Geometer’s Sketchpad, rather than using distance in centimeters to specify the horizontal and vertical components of the vector, units determined by the coordinate plane are used. This coordination is nice and builds on students’ understandings. However, despite the claims of Parnorkou and Maloney, it is not clear what students will come to understand about vectors as having magnitude and direction, or what they will understand about distance when negative values are entered. For example, in excerpt 1 of students’ work with the tool, the researcher asked students to move a square eight spaces to the left. The student stated that, “minus means go down” but he entered –8 to move left, not down. He seemed to understand “go down” as “back or take away.” The use of the negative sign in this manner could be problematic and may not support students’ understanding of the magnitude or direction of a translation vector. Negative values are not used to describe magnitude nor are they used to describe direction. Perhaps a task could be designed that places vectors on a coordinate grid to develop students’ understandings of translation vectors. Several comments about the design of tasks are provided.

Representing vectors
« 6 » ‘To build on students’ understanding of translations as moving in a vertical and horizontal direction, a vector described with coordinates for its head and tail might be introduced (Figure 2).

« 7 » This representation is related to the number line and could move students to thinking about direction up, down, left, and right without using the negative sign to refer to the left and right direction. Another option would be for the dialog box to have four quadrants to allow students to move down and left.

Selection of objects
« 8 » It is also important to note that the objects students are provided to transform are important to consider. Students may
Body Syntonicity in Multi-Point Rotation?

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Referent points

« 9 » It is also imperative that students become familiar with the language of preimage and image points and that students are provided with corresponding points. It was clear that students learned that the “points had to match.” Excerpt 3 shows how students were able to match the two stars, but had difficulty matching the two houses. This is important to note that point A was marked on the preimage and point B on the image was not marked. Students might have assumed point B was located where it was labeled. It is also interesting to note that the preimage (the house on the right) included its own axis but this was not included on the house on the left. These differences may have contributed to students’ difficulty. It is unclear whether points A and B were intended to be corresponding preimage and image points or perhaps they were intended to be labels on the objects. This is a critical distinction that would need to be discussed with students. Also noteworthy is that the sequence of actions students performed on the house task might be reflective of a reactive strategy, rather than a proactive strategy they seemed to use with the star task (Hollebrands 2007). Perhaps the difference was related to the direction of the translation, the object provided, or the interactions the students had with the researcher.

Body Syntonicity in Multi-Point Rotation?

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> Upshot • Parnorkou and Maloney’s article presents an interesting, well-structured and clearly described study of children’s reasoning about mental rotations. Specifically, Parnorkou and Maloney deploy the microworld Graphs ‘n Glyphs, and use it as a “window on thinking-in-change” as they observe and interview children who use it. Reading the article raised a few questions for me about the role of body syntonicity in learning about rotation of geometric shapes, and I wonder where (or if) the authors feel these foundational concepts fit in with their research.

« 1 » The target article presents a study on the development of children’s geometric transformations while using Graphs ‘n Glyphs (GnG). Previous research has primarily focused on quantitative assessment of children’s scores on GT activity tests, and has been done with older learners. The authors argue that the quantitative approach fails to capture not only qualitative elements of this thinking, but also the qualitative properties of the change in this thinking. The study borrows from Richard Noss and Celia Hoyles and uses activities designed in microworld as a “window on thinking-in-change”-approach to understanding how children develop their thinking about geometric transformations. The study then recounts how prior research helped inform them of the design of activities, focusing on translation and rotation (and scale and stretch, not in this article), that would be able to engage younger learners at an appropriate level. Finally, they specify their research questions, focusing in part on how young learners think with GnG, and in part how GnG functions as a research tool for studying this particular kind of thinking.

« 2 » The rest of the article describes first their study, and then a set of excerpts from the researchers’ interactions with two students, and finally addresses their two focused research questions. First, let me start by saying that I think that it is an interesting study and article and that it addresses some central research questions in constructionism. First, how do we design microworlds that simultaneously act as inspiring and fun learning environments and as elicitors of externalized thinking? Straddling the combined agenda of furthering both children’s understanding and our understanding of their understanding is an ever-present challenge in constructionist research and design. The authors do this well by having students solve tasks that encourage explicit dialogue about the hows and whys of solving each of the problems. Second, since we view knowledge as constructed from prior knowledge through reflection on the construction (and manipulation) of shared objects, how do we design microworlds that provide learners with these objects-to-think-with and with relevant primitives for performing mental, physical, or computational operations on them? The combination of the design of the particular activities and GnG seemed to provide students with meaningful ways of discussing and performing the manipulation of shapes.

« 3 » However, two things about the design of primitives that stood out to me were the decisions to, first, have a default direction in rotation (counter-clockwise) and to use minus as “the other way,” and, second, to use + and – in translation as meaning respectively up and down or right or left. This is somewhat of a departure from Logo (and later Logo variants) and their body-syntonic approach to encouraging learners to think about objects as having a “heading” and being able to turn left or right and move forward and backward. Clearly the present design was meaningful to students, and it even

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activated a brief discussion about the nature of minus in which one student said that minus is anti and that “[...] anti means ‘not’” (§31). So I do not think this is a weakness in the design, and in fact one of the strengths of the current design is that it encourages thinking about coordinates in a way that body syntonicity does not necessarily do. A main distinction between Logo turtles and geometric shapes is that geometric shapes consist of many points, whereas Logo turtles are one point each. But since the study showed that the two learners were able to perform rather sophisticated rotation, even around pivot points that were outside of the shape, I do wonder: Would have taking a body syntonic approach to designing these primitives activated other knowledge in students? Would it have affected the way in which the authors’ GT tasks would be designed? Has a body syntonic approach a role in GT learning?

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Authors’ Response: Planting Seeds of Mathematical Abstraction
Nicole Panorkou & Alan Maloney

> Upshot • We consider that elementary students’ situated activities with geometric transformations and animation contain the seeds of complex, and eventually, mathematically generalizable and abstract reasoning. Further studies can explore such technologically-based activities’ potential as building blocks for flexible, creative, and formalized knowledge.

1 We note that a common theme of the commentaries on our article is how we measure, capture, or examine learning in constructionism. In our article, we describe students’ learning of geometric transformations (GT) by studying the changes of their thinking in terms of situated abstractions (Noss & Hoyles 1996). Abstract mathematical ideas of GT were embedded in the tasks and the tools offered to students, and our study was interested in finding how students articulated experiences of GT in the context of graphically-based figures and animation, and developed more abstract or general mathematical reasoning related to that graphical and animation activity.

2 The design of Graphs ’ n Glyphs (GnG) was intended to engage students in fun and productive work in an environment with which students were already familiar – even immersed in – while revealing, and supporting use of, the underlying mathematics of animation. The software was designed for situated activity – computer-based animation – in which increasingly abstracted mathematical/geometrical reasoning become both accessible to students and necessary for sense-making and designing. Students would also build relevant skills: the tool incorporated features of professional animation software, such as dual coordinate planes (not highlighted in this study) and the animation sequencer. Most importantly, the software and associated tasks require students routinely to make direct bidirectional connections between animation design/behavior and more abstracted mathematical reasoning.

3 The students in this study engaged with a combination of constructs earlier than is typical in American elementary education. They gained experience in acting on geometric objects on a coordinate plane, constructed their own descriptions to distinguish among transformation types, and began to adopt precise language to characterize transformations. We consider students’ situated abstractions as having the potential to grow into something more complex and, eventually, mathematically abstract:

[...] the situated, the activity-based, the experiential can contain within it the seeds of something more general. The corollary is that we need to focus on the issue of representation of mathematical objects and how these are expressed; and more fundamentally, how the resources of a setting mediate that expression. (Noss & Hoyles 1996: 49)

4 The commentaries questioned whether students “understood” the abstract ideas we describe (Janet Bowers and Karen Hollebrands), and suggested that there is no indication that the two students had the same understandings about GT (Bowers). We appreciate the opportunity to clarify. In identifying the connections to abstract mathematics, we did not expect the young students to learn in any detail the mathematical notions identified. We intended instead to suggest the potential for their articulations to be leveraged as situated accounts of mathematical experiences, which could be recognized by researchers or teachers enculturated in those mathematical ideas. Also, our goal was not to suggest that all the students had the same understandings. So, for instance, we did not intend to claim that these young students understand translations to be based on vectors or other formal aspects of transformations, but rather to suggest possible connections between the students’ articulations and the mathematical constructs. It remains a matter of conjecture whether such experiences might subsequently be building blocks for such formalized knowledge, but the students’ work provides an expanded vision of what elementary-age students are capable of learning about GT when tasks and software give them the opportunities to do so.

5 The nature of such teaching experiments, especially using an innovation such as a technological tool, is to engineer the conditions of learning (the learning ecology) and explore the changes in student learning that result – through student work products, discourse, and classroom interac-
tions. Our interpretation of potential mathematical abstraction (e.g., the abstractions we connected to Nate’s analogy of different positions of the center of rotation, relative to an object and to the Earth’s rotation and orbit) would perhaps be more accurately represented as a conjecture or “humble theory” (Cobb et al. 2003; Confrey & Maloney 2015) that can now be incorporated into instructional goals as a basis of a subsequent teaching experiment.

« 6 » Three different teaching experiments using GnG (this study, Confrey et al. 2010, and one not yet published) have now seen students (fourth through eighth grades) construct figures and animations while developing and using mathematical language and concepts that are rooted in the activities of transformations. The aforementioned bidirectionality between situated context and more abstracted mathematical/geometrical reasoning is a consistent feature of students’ work with the tool and tasks, suggesting that by linking the situated context with the geometric space in which transformations are enacted, GnG and tasks dynamically support the advancement of students’ thinking about GT.

« 7 » Arthur Hjorth raised the question of designing microworlds that are inspiring and fun learning environments while eliciting externalized thinking (an explicit goal of GnG). GnG mimics some aspects of game environments, setting challenges for students within a microworld whose rules are readily discerned, not overly restrictive, novel for mathematics instructional contexts, and closely related to the experts’ tools. And it has avoided automated or drag-and-drop tools, which would obscure the underlying mathematical basis of the animations.

« 8 » Bowers suggested more creative use of the animation component of the software. In the current study, we did not extensively employ development of animation sequences (the sequencer was designed “to support multiple steps in one execution” as well as “part of the larger goal of learning animation”), but this would be a focus of a more extended study. We note that in a previous study (Confrey et al. 2010), middle-grade students created highly complex and inventive animations, such as schools of fish moving in the sea and a motorcycle doing “wheelies,” and contributed to design improvement in the software as well. The richness of transformation sequences and student collaboration is practically boundless and highly engaging for the students.

« 9 » The commenters suggested several modifications, which offer potential strengthening of connections between the animation context and explicit geometrical and numerical-operational abstraction. We are considering: a vector tool to support students to generalize diagonal translations; on-demand availability of all four quadrants on local and global coordinate planes (students identify the need for negative coordinate values) and to support (per Hollebrands) students’ making meaning for, and distinguishing between, negative values for direction and position; more “real-life” images; and enriched tools for creating new shapes.

« 10 » By exploring transformations, elementary students can learn foundational, if non-formal, versions of GT concepts, and make connections to related geometric concepts (properties of shapes, mapping, magnitude, direction, ratio, and negative numbers). We see these seeds for development of such notions for later years of schooling as similar to Seymour Papert’s (1980) “gears” of his childhood, early experiences with the gearbox and the differential that served as models that set the stage for rapidly making sense of equations:

**I found particular pleasure in such systems as the differential gear, which does not follow a simple linear chain of causality since the motion in the transmission shaft can be distributed in many different ways to the two wheels depending on what resistance they encounter [.]. I saw multiplication tables as gears, and my first brush with equations in two variables (e.g., 3x + 4y = 10) immediately evoked the differential. By the time I had made a mental gear model of the relation between x and y, figuring how many teeth each gear needed, the equation had become a comfortable friend.** (Papert 1980: vi–vii)

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Combined References


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